Indian Statistical Institute, Bangalore
B. Math.
First Year,
Analysis I

Mid-term Supplementary examination — Date : Nov. 25, 2019
Total Marks: 105 Maximum marks: 60 — Time: 3 hours

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$-the set of natural numbers. $\mathbb{R}$-the set of real numbers.

(1) A subset D of $\mathbb{R}^2$ is called an open disc if D is of the form:

$$D = \{(x, y) : (x - a)^2 + (y - b)^2 < r\}$$

for some $(a, b) \in \mathbb{R}^2$ and some $r > 0$. Two open discs D_1 and D_2 are disjoint if $D_1 \bigcap D_2$ is the empty set. Suppose A is a set of disjoint open discs. Show that A is countable. [15]

(2) Show that there exists a real number x such that $x^2 = 3$. [15]

(3) Let M be a subset of $[0, 1)$ such that 0 is the only limit point of M. Show that M is countable and there exists a decreasing sequence $\{m_n\}_{n\geq 1}$ such that

$$M = \{m_n : n \in \mathbb{N}\}.$$

[15]

(4) Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers with

$$\lim_{n\to\infty} a_n = 2, \ \lim_{n\to\infty} b_n = 5.$$

Define c_n for $n \geq 1$, by

$$c_n = \text{ Max } \{a_n, b_n\}.$$

Show that $\{c_n\}_{n\geq 1}$ is a convergent sequence and $\lim_{n\to\infty} c_n = 5$. [15]

(5) Show that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. [15]

(6) Let $\{x_n\}_{n\geq 1}$, and $\{y_n\}_{n\geq 1}$ be bounded sequences of real numbers. Show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

Give an example where the equality does not hold. [15]

(7) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. [15]